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Development of the boundary layer at a free surface from a uniform shear flow

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The development of the boundary layer accompanying the formation of a free surface at $y' = 0$, from the two-dimensional uniform shear flow $u' = \omega y'$, is discussed. The analysis shows that the surface velocity and surface position vary as the cube root of the distance downstream, while the mass-transfer coefficient varies inversely as the cube root of this distance. It is shown how these may be applied to the formation of capillary jets.

1. Introduction

The cardinal hydrodynamic feature in the formation of a new liquid surface when a liquid stream separates from a solid is the instantaneous removal of the wall shear-stress. Near the point where the surface is formed, at the large Reynolds numbers at least, the shear stress must fall from some finite value, at a point slightly within the liquid, to zero at the free surface. At the same time, the velocity of a fluid element on the surface increases from zero with distance downstream. In this paper we shall present a momentum- and mass-transfer analysis of the circumstances accompanying the formation of a free surface (at $y' = 0$) from the two-dimensional uniform shear flow, $u' = \omega y'$, parallel to the x'-axis. The flow is taken to be of infinite extent in both the y'- and z'-directions and gravitational and surface-tension forces are neglected. The liquid is assumed to be Newtonian with constant physical properties. Besides being the simplest case to analyze, this is also one of practical interest because it approximates the conditions in the formation of capillary jets and in some other flow systems for gas-liquid contacting. As one might expect from the absence of both a characteristic length and a characteristic velocity, the problem admits a similarity solution. The analysis shows that the surface velocity and surface position vary as the cube root of the distance downstream, while the local mass-transfer coefficient varies inversely as the cube root of this distance.

The most closely related published work concerns the development of the boundary layer over a flat plate from the uniform shear flow, $u' = U_0 + \omega y'$, parallel to the x'-axis. In a recent discussion of this problem, Ting (1960) pointed out that in the earlier investigations of Li (1955, 1956, 1957), Glauert (1957) and Murray (1961) the effect of the shear was obtained as a small perturbation. He then went on to solve the problem when the shear, $\omega y'$, is the major effect and the uniform velocity U_0 can be treated as a perturbation. This is precisely the condition in the present problem, i.e. $U_0 = 0$. The present problem is made somewhat more complicated however by the fact that the surface is free to change and its position must be calculated. There seems to be some controversy as to whether the viscous flow in the boundary layer over the flat plate induces a pressure in the inviscid flow outside of the boundary layer. The analyses of Li, Murray and Ting indicate that such a pressure is induced, but Glauert argued against the existence of an induced pressure. In the present work it would seem that the free surface would preclude the existence of such a pressure (to the usual order of magnitude of boundary-layer analyses), at least when gravitational and surface-tension forces are neglected.

Another closely related work is that of Scriven & Pigford (1959). Following a suggestion of Rideal & Sutherland (1952), they adapted Goldstein's (1930, 1933) boundary-layer analysis—for the joining of the two streams being shed from a flat plate of finite length placed in a uniform flow—to predict the change in surface velocity of a capillary jet issuing from an orifice. For their situation the initial velocity profile deviates from plug flow only in a narrow peripheral zone of the jet, and so any change in average jet velocity or radius is small. On the other hand, for a capillary jet issuing from a long needle in which parabolic flow is established, the relaxation of the shear from a large portion of the jet results in appreciable changes in average jet velocity and radius.

Other papers have discussed the simultaneous development of momentum and concentration boundary layers when two different parallel streams in plug flow meet. See, for example, the works of Lock (1951) and Potter (1957).

2. Momentum transfer

Consider the flow situation sketched in figure 1. The two-dimensional, uniform shear flow, $\omega y'$, of a Newtonian liquid with constant physical properties is caused to leave the solid wall ($y' = 0$) and contact an immiscible and inviscid medium. Surface-tension and gravitational forces will be neglected. At the high Reynolds numbers it seems likely that the effects of the sudden removal of the viscous shear stress at the wall are confined to a thin, but spatially growing region adjacent to the free surface. One would therefore anticipate that in this region derivatives with respect to y are much larger than derivatives with respect to x and the x-component of velocity is much larger than the y-component. If the curvature of the free surface is small, then the development of the flow is described by the usual two-dimensional boundary-layer equations as derived in Goldstein (1938):

$$\frac{\partial u}{\partial x}+\frac{\partial y}{\partial v} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}. \tag{2}$$

We have omitted the derivative of the pressure in the x-momentum equation because along the free surface the pressure is 'constant', and since an order-of-magnitude analysis on the y-momentum equation shows the y-dependence of the pressure to be of second order, the pressure may be taken as constant everywhere.

For a more complete discussion of why the pressure does not appear in (2), see the appendix.

Near the free surface the component of the uniform shear in the x-direction can be approximated by $\omega(\zeta + y\cos\phi)\cos\phi \approx \omega(\zeta + y)$, where ζ is the local displacement of the free surface. The x-component of the velocity will be written as $u = \omega(\zeta + y) + \hat{u}$. A little investigation shows that the problem permits a similarity

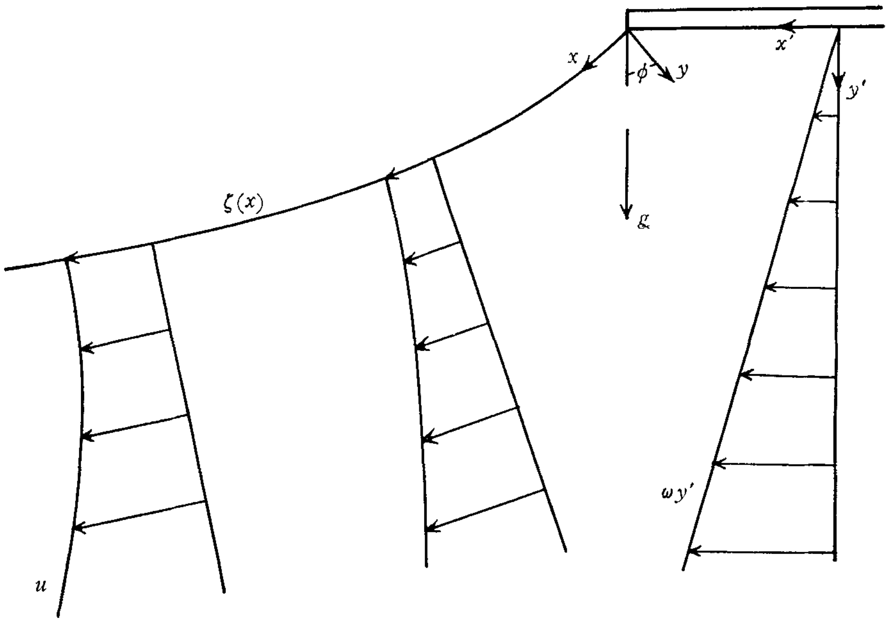

FIGURE 1. Geometry of the flow system.

solution if ζ and $\hat{u}$ are of the form $\zeta = Cx^{\frac{1}{3}}$ and $\hat{u} = \omega A x^{\frac{1}{3}} g(\eta)$, where $\eta = By/x^{\frac{1}{3}}$. Two of the constants, A and B, may be chosen arbitrarily, but the third is determined by the mathematical nature of the problem and must be consistent with the constraint of constant volumetric flow rate. It turns out to be convenient to choose the constants so that

$$\zeta = \beta(3\nu x/\omega)^{\frac{1}{3}}, \tag{3}$$

and

$$u = (3\nu x\omega^2)^{\frac{1}{3}}\{\beta + \eta + g(\eta)\}, \tag{4}$$

where

$$\eta = (\omega y^3/3\nu x)^{\frac{1}{3}}, \tag{5}$$

and $\beta = C/A$ is a constant. Substituting these expressions into the continuity equation and then integrating with respect to y yields the following expression for the y-component of velocity:

$$v = -(\omega\nu^2/3x)^{\frac{1}{3}}\left\{\beta\eta + 2\int_0^\eta g\,d\eta - \eta g\right\}. \tag{6}$$

When (4), (5) and (6) are substituted into the x-momentum equation it gives the following ordinary differential equation for $g(\eta)$:

$$g'' = \{\beta + \eta + g\}\{\beta + g - \eta g'\} - \{1 + g'\}\left\{\beta\eta + 2\int_0^\eta g\,d\eta - \eta g\right\}. \tag{7}$$

Two boundary conditions for this equation derive from the physical conditions that the viscous shear stress vanishes at the free surface ($y = 0$) and the modification to the uniform shear velocity profile vanishes infinitely far from this surface. These take the form

$$g(0) = -1, \tag{8}$$

and

$$g(\infty) = 0. \tag{9}$$

In addition, the constraint of constant volumetric flow rate must be satisfied. This condition is found by equating the flow (with velocity $\omega y'$) lost due to the change in surface position, namely $\int_{-\zeta}^{0} \omega(\zeta + y)\,dy$, to the flow gained by the modification of the velocity profile, namely $\int_0^\infty \hat{u}\,dy$. In terms of the similarity function $g(\eta)$ the restriction can be written as

$$\beta^2 = 2\int_0^\infty g\,d\eta. \tag{10}$$

It can be shown that the same requirement is needed for the boundary-layer solution to satisfy Euler's equation with zero pressure gradient infinitely far from the free surface.

The method we have used to solve this problem is that of Meksyn (1961). We first seek a power-series representation for $g(\eta)$. The boundary condition at infinity is then expressed in terms of definite integrals which are evaluated by the method of steepest descent. If we denote by α the value of g at $\eta = 0$, then the value of the second derivative at $\eta = 0$ may be found from (7). Thus we see

$$g(0) = \alpha, \quad g'(0) = -1, \quad \text{and} \quad g''(0) = (\alpha+\beta)^2. \tag{11}$$

The third derivative is found by differentiating (7), giving

$$g''' = -g''\left\{2\beta\eta + \eta^2 + 2\int_0^\eta g\,d\eta\right\}, \tag{12}$$

and we see

$$g'''(0) = 0. \tag{13}$$

The values of the higher derivatives are found by successive differentiation. In this way we find

$$\left.\begin{aligned} &g^{\text{iv}}(0) = -2(\alpha+\beta)^3, && g^{\text{v}}(0) = 0, \\ &g^{\text{vi}}(0) = 10(\alpha+\beta)^4, && g^{\text{vii}}(0) = 0, \\ &g^{\text{viii}}(0) = -56(\alpha+\beta)^5, && g^{\text{ix}}(0) = 0, \ldots \end{aligned}\right\} \tag{14}$$

Thus $g(\eta)$ is given by the power series

$$g(\eta) = \alpha - \eta + \frac{(\alpha+\beta)^2\eta^2}{2!} - \frac{2(\alpha+\beta)^3\eta^4}{4!} + \frac{10(\alpha+\beta)^4\eta^6}{6!} - \frac{56(\alpha+\beta)^5\eta^8}{8!} + \ldots. \tag{15}$$

We evaluate the constants α and β in the following way. With the known value of $g''(0)$, (12) can be integrated formally to give

$$g'' = (\alpha+\beta)^2 e^{-G(\eta)}, \tag{16}$$

where

$$G(\eta) = \int_0^\eta \left\{2\beta\eta + \eta^2 + 2\int_0^\eta g\,d\eta\right\} d\eta. \tag{17}$$

When the power-series expansion for $g(\eta)$ is substituted into (17) and the indicated integrations are performed, the following series expression for $G(\eta)$ is obtained:

$$G(\eta) = \xi^2 + \frac{2}{4!}\xi^4 - \frac{4}{6!}\xi^6 + \frac{20}{8!}\xi^8 - \frac{112}{10!}\xi^{10} + \ldots, \tag{18}$$

where

$$\xi = (\alpha+\beta)^{\frac{1}{2}}\eta. \tag{19}$$

Two successive integrations of (16) from $\eta = 0$ to η give

$$g(\eta) = \left\{\alpha - (\alpha+\beta)^2 \int_0^\eta \eta e^{-G} d\eta\right\} - \eta\left\{1 - (\alpha+\beta)^2 \int_0^\eta e^{-G} d\eta\right\}, \tag{20}$$

in which integration by parts has been used to simplify the result. The motivation here is that $G(\eta)$ is a rapidly increasing function of η so that e^{-G} is a rapidly decreasing function and contributions to integrals such as in (20) come only from the region of small η where the power-series representation (18) is known and accurate. Another integration of (20) followed by some rearrangement gives

$$\int_0^\eta g\,d\eta = \tfrac{1}{2}(\alpha+\beta)^2 \int_0^\eta \eta^2 e^{-G} d\eta + \eta\left\{\alpha - (\alpha+\beta)^2 \int_0^\eta \eta e^{-G} d\eta\right\} - \tfrac{1}{2}\eta^2\left\{1 - (\alpha+\beta)^2 \int_0^\eta e^{-G} d\eta\right\}. \tag{21}$$

A comparison of this equation with (10) shows that in order for the constraint of constant volumetric flow rate to be satisfied we must have

$$\left.\begin{aligned} 1 &= (\alpha+\beta)^2 \int_0^\infty e^{-G} d\eta = (\alpha+\beta)^{\frac{3}{2}} \int_0^\infty e^{-G} d\xi = (\alpha+\beta)^{\frac{3}{2}} I, \\ \alpha &= (\alpha+\beta)^2 \int_0^\infty \eta e^{-G} d\eta = (\alpha+\beta) \int_0^\infty \xi e^{-G} d\xi = (\alpha+\beta) J, \\ \beta^2 &= (\alpha+\beta)^2 \int_0^\infty \eta^2 e^{-G} d\eta = (\alpha+\beta)^{\frac{1}{2}} \int_0^\infty \xi^2 e^{-G} d\xi = (\alpha+\beta)^{\frac{1}{2}} K. \end{aligned}\right\} \tag{22}$$

It will be noted that these relations also satisfy the condition that $g(\infty) = 0$. See equation (20). At first it appears that the system of three equations for two unknowns is overdetermined. The equations will have a non-contradictory solution only if the relation $(I-J)^2 = IK$ holds. The validity of this relation is confirmed by the numerical values of the integrals. Its validity probably could be established by analytical means, but this has not been done.

Numerical evaluation of the integrals is accomplished by the method of steepest descent. To consider the integral K, for example, we first change the variable of integration from ξ to G:

$$K = \int_0^\infty \xi^2 e^{-G} d\xi = \int_0^\infty e^{-G}\left(\xi^2 \frac{d\xi}{dG}\right) dG. \tag{23}$$

Inverting the series expansion for G, (18), we find

$$\xi = G^{\frac{1}{2}}\{1 - 0{\cdot}041667G + 0{\cdot}0088542G^2 - 0{\cdot}0024833G^3 + 0{\cdot}00078574G^4 \ldots\}, \tag{24}$$

and when this is substituted into (23) we obtain

$$K = \frac{1}{2}\int_0^\infty e^{-G}G^{\frac{1}{2}}\{1-0\cdot20833G+0\cdot074132G^2-0\cdot029207G^3+0\cdot011951G^4-\ldots\}\,dG. \tag{25}$$

The integral is evaluated term by term. The series obtained is found not to be converging (as indicated by the first five terms) which reflects the fact that the series for $g(\eta)$ has a finite radius of convergence. To convert this series to a convergent one the Euler transformation (see Meksyn 1961) is applied to the terms from three onwards. A second Euler transformation is then applied to the terms from five onwards. This is indicated in the following scheme:

$$\begin{aligned} K &= \tfrac{1}{4}\sqrt{\pi}\{1-0\cdot31250+0\cdot27799-0\cdot38334+0\cdot70585-\ldots\} \\ &= \tfrac{1}{4}\sqrt{\pi}\{1-0\cdot31250+0\cdot13900-0\cdot02634+0\cdot02715-\ldots\} \\ &= \tfrac{1}{4}\sqrt{\pi}\{1-0\cdot31250+0\cdot13900-0\cdot02634+0\cdot01628-\ldots\} \\ &= \tfrac{1}{4}\sqrt{\pi}\times 0\cdot808 = 0\cdot358. \end{aligned} \tag{26}$$

The value adopted is the average of the fourth and fifth approximations because in view of the alternating signs it is thought to give a slightly better approximation. A similar treatment of the integrals I and J gives

$$\left.\begin{aligned} I &= \tfrac{1}{2}\sqrt{\pi}\times 0\cdot956 = 0\cdot847, \\ \text{and}\qquad J &= \tfrac{1}{2}\times 0\cdot888 = 0\cdot444. \end{aligned}\right\} \tag{27}$$

We are now in a position to certify the relation $(I-J)^2 = IK$. By inserting the above values we find $(I-J)^2/IK$ to be 1·02, which confirms the relation and implies that the numerical values are accurate to about 2 %. With these values of the integrals, the constants α and β are easily computed and are found to be 0·491 and 0·615 respectively. Thus, the surface position and surface velocity are given by the formulae

$$\left.\begin{aligned} \zeta &= 0\cdot615(3\nu x/\omega)^{\frac{1}{3}}, \\ \text{and}\qquad u_{\text{surf}} &= 1\cdot106(3\nu x\omega^2)^{\frac{1}{3}}. \end{aligned}\right\} \tag{28}$$

The modification to the uniform shear velocity profile can now be calculated from (15) for small η or from (20) for large η.

3. Mass transfer

Newly formed liquid surfaces are frequently encountered under conditions where there is transfer of a chemical species (or heat) from a gas phase to a liquid phase. It is therefore of interest to carry out the calculation for the rate of mass transfer in the present flow system. We have examined the situation for which the liquid is initially devoid of a chemical species and there is no resistance to transfer in the gas phase, i.e. the surface is maintained at a fixed concentration c_s. With the boundary-layer approximations the equation for the conservation of the species reads

$$u\frac{\partial c}{\partial x}+v\frac{\partial c}{\partial y} = D\frac{\partial^2 c}{\partial y^2}, \tag{29}$$

and the boundary conditions are

$$\left.\begin{aligned} &\text{at} \quad y = 0, \quad c = c_s \\ \text{and} \quad &\text{at} \quad y = \infty \quad \text{or} \quad x = 0, \quad c = 0. \end{aligned}\right\} \tag{30}$$

If the concentration profile is of the form $c = c_s h(\eta)$ and if this and the velocities obtained in the previous section are substituted into the conservation equation, the following ordinary differential equation for $h(\eta)$ is obtained:

$$h'' = -h' \, Sc\left\{2\beta\eta + \eta^2 + 2\int_0^\eta g\,d\eta\right\}, \tag{31}$$

where $Sc = \nu/D$ is the Schmidt number. The boundary conditions become

$$h(0) = 1, \quad h(\infty) = 0. \tag{32}$$

Equation (31) is easily solved subject to the conditions in (32) and the result is

$$h(\eta) = 1 - \left\{\int_0^\eta e^{-ScG}\,d\eta \bigg/ \int_0^\infty e^{-ScG}\,d\eta\right\}. \tag{33}$$

To calculate the local mass-transfer coefficient we compute the flux of species through the surface by molecular diffusion and equate this to the product of the mass-transfer coefficient and the concentration driving force.

$$k \equiv -D\frac{\partial h}{\partial y}\bigg|_{y=0} = D(\omega/3\nu x)^{\frac{1}{3}} \bigg/ \int_0^\infty e^{-ScG}\,d\eta. \tag{34}$$

The average mass-transfer coefficient is found by integration:

$$k_{\text{avg}} \equiv \frac{1}{x}\int_0^x k\,dx = \tfrac{3}{2}\,D(\omega/3\nu x)^{\frac{1}{3}} \bigg/ \int_0^\infty e^{-ScG}\,d\eta. \tag{35}$$

The only remaining task is to evaluate the integral in these expressions. This is accomplished as above by the method of steepest descent with the result that

$$\int_0^\infty e^{-ScG}\,d\eta = \frac{\sqrt{\pi}}{2(\alpha+\beta)^{\frac{1}{2}}\,Sc^{\frac{1}{2}}}\left\{1 - \frac{0{\cdot}062500}{Sc} + \frac{0{\cdot}033203}{Sc^2} - \frac{0{\cdot}032593}{Sc^3} + \frac{0{\cdot}046408}{Sc^4} - \dots\right\}. \tag{36}$$

For Schmidt numbers greater than unity the integral is closely approximated by $\frac{1}{2}\sqrt{\pi}(\alpha+\beta)^{-\frac{1}{2}}Sc^{-\frac{1}{2}}$. Even at $Sc = 1$ this approximation overestimates the integral only by about 5%. With this approximation the mass-transfer coefficients become

$$\left.\begin{aligned} k &= 1{\cdot}19 D\,Sc^{\frac{1}{2}}(\omega/3\nu x)^{\frac{1}{3}} \\ \text{and} \qquad k_{\text{avg}} &= 1{\cdot}79 D\,Sc^{\frac{1}{2}}(\omega/3\nu x)^{\frac{1}{3}}. \end{aligned}\right\} \tag{37}$$

4. Application to capillary jets

The results derived above are applicable to newly formed capillary jets for small axial distances provided the interaction of the boundary layer with the core region of the jet, where the initial velocity profile is not one of uniform shear, is negligible. For a jet of average velocity $\overline{w}_0$ issuing from a long needle of radius a_0 with parabolic flow, the shear rate at the wall is $4\overline{w}_0/a_0$. Taking $\omega = 4\overline{w}_0/a_0$ we find that (28) and (37) give the following expressions for the initial change in jet radius, the surface velocity, and the average mass-transfer coefficient:

$$\left.\begin{aligned} \zeta/a_0 &= 0{\cdot}703(x/a_0 Re)^{\frac{1}{3}}, \\ u_{\text{surf}}/\overline{w}_0 &= 5{\cdot}07(x/a_0 Re)^{\frac{1}{3}}, \\ k_{\text{avg}}a_0/D &= 1{\cdot}56 Sc^{\frac{1}{2}}(x/a_0 Re)^{-\frac{1}{3}}, \\ Re &= 2a_0\overline{w}_0/\nu. \end{aligned}\right\} \tag{38}$$

where

In another paper, Goren & Wronski (1966) reported measurements of the radius of capillary jets as a function of axial distance. For their highest Reynolds numbers, about 200, the measurements confirmed the cube-root dependence on the axis distance, but the observed coefficient was lower than the predicted one by about a factor of 2. Whether the discrepancy between the theory and experiment is due to the smallness of the Reynolds number or to the interaction of the peripheral boundary layer and the core is, at present, unknown.

Appendix: The momentum equation

Take the x and y components of velocity as $\omega(\zeta+y)+\hat{u}$ and $\hat{v}$ respectively, and assume the boundary-layer quantities are functions of y/δ and x/L, where δ is a measure of the boundary-layer thickness, L is a measure of the distance downstream, and $\delta/L \ll 1$. The boundary condition that the shear stress vanishes at the free surface requires that the order of magnitude of $\hat{u}$ be $\omega\delta$, while the continuity equation requires that the order of magnitude of $\hat{v}$ be $\omega\delta^2/L$. To the usual boundary-layer order-of-magnitude estimates, the y-momentum equation can be written as

$$\frac{d^2\zeta}{dx^2}u^2 = -\frac{1}{\rho}\frac{\partial p}{\partial y}+g, \tag{39}$$

where $d^2\zeta/dx^2$ is the curvature of the free surface and g is the acceleration of gravity. Integrating this equation gives

$$p = \rho g y - \int_0^y \frac{d^2\zeta}{dx^2}u^2\,dy + p_{\text{surf}}. \tag{40}$$

The value of the pressure at the surface, p_{surf}, is found from the boundary condition that the change in normal stress there is due to the surface tension and the curvature, namely

$$-p_{\text{surf}} + 2\mu\left.\frac{\partial v}{\partial y}\right|_{y=0} = -\sigma\frac{d^2\zeta}{dx^2}. \tag{41}$$

Equation (40) becomes

$$p = \rho g y - \int_0^y \frac{d^2\zeta}{dx^2}u^2\,dy + 2\mu\left.\frac{\partial v}{\partial y}\right|_{y=0} + \sigma\frac{d^2\zeta}{dx^2}, \tag{42}$$

or upon differentiation

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{1}{\rho}\frac{\partial}{\partial x}\int_0^y \frac{d^2\zeta}{dx^2}u^2\,dy - 2\nu\frac{\partial}{\partial x}\left(\left.\frac{\partial v}{\partial y}\right|_{y=0}\right) - \frac{\sigma}{\rho}\frac{d^3\zeta}{dx^3}. \tag{43}$$

When (43) is substituted into the x-momentum equation and the usual boundary-layer order-of-magnitude estimates are made, the first two terms on the right-hand side of (43) are seen to be negligible with respect to the term $\nu\, \partial^2 u/\partial y^2$. The x-momentum equation may then be written as

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma}{\rho}\frac{d^3\zeta}{dx^3}+g\frac{d\zeta}{dx}. \tag{44}$$

For the inertial and viscous terms to be of the same order of magnitude within the boundary layer we must have $\delta/L \sim (\omega L^2/\nu)^{-\frac{1}{3}}$ and the boundary-layer treatment is valid only if $\omega L^2/\nu \gg 1$. The magnitude of the ratio of the surface-tension term to the inertial term is $(\sigma/\rho\omega^{\frac{1}{3}}\nu^{\frac{2}{3}})\,(\omega L^2/\nu)^{-\frac{7}{6}}$; the magnitude of the ratio of the gravitational term to the inertial term is $(g/\omega^{\frac{3}{2}}\nu^{\frac{1}{2}})\,(\omega L^2/\nu)^{-\frac{1}{6}}$. For the flow of a liquid of moderate viscosity ($\nu = 0{\cdot}1\,\mathrm{cm^2/sec}$) at a high rate of shear ($\omega = 10^4\,\mathrm{sec^{-1}}$), $\delta/L = 0{\cdot}1$ at $L = 0{\cdot}1$ cm. Furthermore, with $\sigma = 30$ dyne/cm and $\rho = 1\,\mathrm{g/cm^3}$, $(\sigma/\rho\omega^{\frac{1}{2}}\nu^{\frac{3}{2}})\,(\omega L^2/\nu)^{-\frac{7}{6}} \approx 3\times 10^{-3}$ and $(g/\omega^{\frac{3}{2}}\nu^{\frac{1}{2}})\,(\omega L^2/\nu)^{-\frac{1}{6}} \approx 10^{-3}$. It would appear that the surface-tension and gravitational forces might be neglected and that the boundary-layer approximations would be valid for distances downstream greater than about 0·1 cm.

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